Optical Theorem and Off-Diagonal Elements of the T Matrix

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The optical theorem requires that the anti-Hermitian part of the T matrix have positive diagonal elements in *any* representation. The condition for this is that the anti-Hermitian part be a positive Hermitian matrix; this imposes restrictions in the form of inequalities, which involve off-diagonal as well as diagonal elements of the matrix in any particular representation. Specific results are derived for "spin-flip" amplitudes in the case where particles have spins $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},1)$, and $(1,1)$. Application to reaction amplitudes is also briefly discussed.

INTRODUCTION

In general, neither the total spin angular momentum, its projection on a given axis, nor the projections of the spins of the individual particles of a physical system are constants of the motion. If one considers the forward elastic scattering of one particle by another there will then exist several amplitudes describing the scattering associated with the various spin configurations of the initial and final states. In particular, we may use a barycentric reference frame in which the momentum $\mathbf{p}$ of the first particle lies along the z axis and use this direction for the direction of spin quantization. Let us designate the spin of the first particle by A and its z projection by a, the spin of the second by B and its z projection by b. In a representation in which the spin projections are diagonal, the various forward elastic-scattering amplitudes can then be written as a matrix f with elements $\langle a'b'|f|ab\rangle$, where a, b are the projections in the initial state and a', b' the projections in the final state. In general, this matrix is neither diagonal nor diagonalizable. The optical theorem tells us, however, that if we split f into its Hermitian and anti-Hermitian parts, $f=g+ih$, then the diagonal elements of h are positive:

$$\langle ab|h|ab\rangle \geqslant 0. \qquad (1)$$

Instead of labeling the initial and final states by the individual spin projections, we could label them by the total spins C and C' and total spin projections c and c' of the initial and final states, respectively. In this representation, which is unitarily related to the preceding one, the matrix f would have elements $\langle C'c'|f|Cc\rangle$, and the optical theorem would require

$$\langle Cc|h|Cc\rangle \geqslant 0. \qquad (2)$$

Conditions (1) and (2) are not necessarily equivalent since the matrix element $\langle Cc|h|Cc\rangle$ involves off-diagonal as well as diagonal matrix elements $\langle a'b'|h|ab\rangle$.

It is therefore natural to inquire what are the strongest conditions imposed by the optical theorem on the matrix elements of h in any given representation. The answer is simply that the associated Hermitian form shall be positive, which in turn requires that the determinant of h and all principal minors of this determinant (which are necessarily real) be positive.[1] What is essentially involved here is the following: The positiveness conditions on the diagonal matrix elements of h are in general representation-dependent. They take their most stringent form, however, in that representation in which h is diagonal, and when they are imposed in that representation they take the above form in any other representation. The resultant inequalities then impose limitations on the magnitude of off-diagonal ("spin-flip") amplitudes in such representations.

In certain representations, including the two described above, invariance under rotations and under time reversal lead to the condition that f is a symmetric matrix, whereupon h is a real symmetric matrix. In such cases the inequalities are then conditions on the imaginary parts of the matrix elements of f.

In the following sections, we apply these results to the cases where $(A,B)=(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},1)$, $(1,1)$—these being the simplest nontrivial cases, in both the (ab) and (Cc) representations. To keep the results simple, we shall assume that the system is invariant under rotations, space inversion, and time reversal, since these circumstances severely limit the number of independent matrix elements of f.

THE INDIVIDUAL SPIN REPRESENTATION

In the (ab) representation in which initial and final spin states are labeled by the spin projections of the individual particles in the direction of $\mathbf{p}$, time-reversal invariance and rotational invariance impose the following conditions on the matrix elements:

$$\langle ab|f|a'b'\rangle = \langle a'b'|f|ab\rangle,$$
$$\langle a'b'|f|ab\rangle = 0 \quad \text{unless} \quad a'+b'=a+b,$$

while invariance under space inversion yields the further conditions

$$\langle -a'-b'|f|-a-b\rangle = \langle a'b'|f|ab\rangle.$$

For simplicity of typography we use Greek letters to label the nonzero (real) elements of the h matrix, but it should be noted that the same Greek letter refers to different matrix elements in the different special cases.

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[1] See, for example, I. M. Gel'fand, *Lectures on Linear Algebra* (Interscience Publishers, Inc., New York, 1961), p. 68.

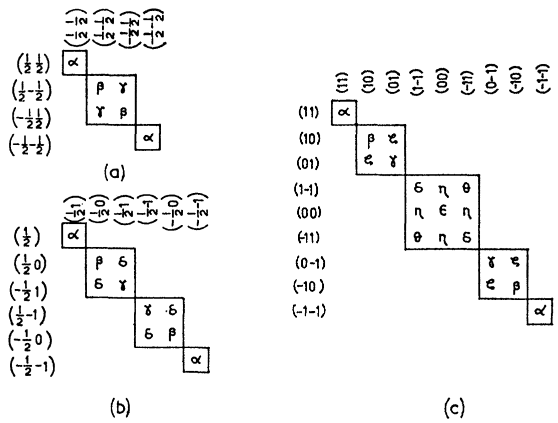

FIG. 1. Matrices in the individual spin representation. The rows are labeled by $(a'b')$ and the columns by (ab). (a) $A=B=\frac{1}{2}$, (b) $A=\frac{1}{2}$, $B=1$; (c) $A=B=1$.

$A=\frac{1}{2}$, $B=\frac{1}{2}$. In this case the h matrix takes the form shown in Fig. 1(a). The independent inequalities arising from the positiveness of the principal minors are

$$\alpha,\beta\geqslant 0,\qquad \beta^2-\gamma^2\geqslant 0. \tag{3}$$

$A=\frac{1}{2}$, $B=1$. The h matrix for this case is shown in Fig. 1(b). The resultant independent inequalities are

$$\alpha,\beta\geqslant 0,\qquad \beta\gamma-\delta^2\geqslant 0,$$

from which can be derived the further inequality

$$\gamma\geqslant 0.$$

$A=1$, $B=1$. The h matrix is given in Fig. 1(c) and the resultant independent inequalities are

$$\alpha,\beta,\delta\geqslant 0,$$
$$\beta\gamma-\zeta^2\geqslant 0,\quad \delta\epsilon-\eta^2\geqslant 0,$$
$$(\delta-\theta)[(\delta+\theta)\epsilon-2\eta^2]\geqslant 0.$$

From these can be derived the further inequalities

$$\gamma,\epsilon\geqslant 0,\quad \delta\geqslant\theta,$$
$$\tfrac{1}{2}(\delta+\theta)\epsilon-\eta^2\geqslant 0.$$

The above results are also valid in a helicity representation which is obtained simply from that above by the substitutions: $a\rightarrow\eta_A$, $b\rightarrow -\eta_B$, $a'\rightarrow\eta'_A$, $b'\rightarrow -\eta'_B$, where $\eta_{A,B}$ and $\eta_{A,B}'$ represent the initial and final helicities of the particles.

THE TOTAL-SPIN REPRESENTATION

In the (Cc) representation in which the total spin and its projection on the direction of $\mathbf{p}$ are diagonal one has the following conditions on the matrix elements of f as a result of invariance under rotations, time reversal, and space inversion:

$$\langle C'c'|f|Cc\rangle=\langle Cc|f|C'c'\rangle,$$
$$\langle C'c'|f|Cc\rangle=0\quad\text{unless}\quad c'=c,$$
$$\langle C'-c'|f|C-c\rangle=(-1)^{C'-C}\langle C'c'|f|Cc\rangle.$$

We examine the same special cases but with the omission of $A=\frac{1}{2}$, $B=\frac{1}{2}$, since for this case the f matrix is diagonal in this representation.

$A=\frac{1}{2}$, $B=1$. The h matrix for this case is given in Fig. 2(a), with the resultant independent inequalities,

$$\alpha,\beta\geqslant 0,\qquad \beta\gamma-\delta^2\geqslant 0,$$

and the derived inequality,

$$\gamma\geqslant 0.$$

$A=1$, $B=1$. The h matrix for this case is given in Fig. 2(b) with the independent inequalities taking the form

$$\alpha,\beta,\delta,\epsilon\geqslant 0,$$
$$\beta\gamma-\eta^2\geqslant 0,\quad \delta\zeta-\theta^2\geqslant 0,$$

from which follow the further inequalities,

$$\gamma,\zeta\geqslant 0.$$

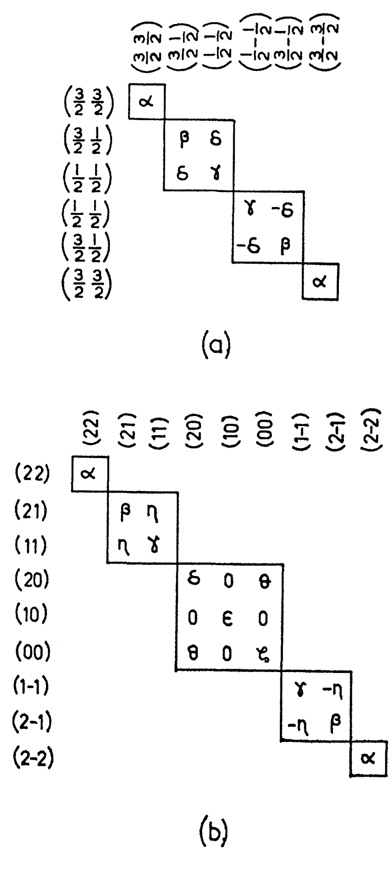

FIG. 2. Matrices in the total spin representation. the rows are labeled by $(C'c')$ and the columns by (Cc). (a) $A=\frac{1}{2}$, $B=1$. (b) $A=B=1$.

AN APPLICATION

To explore the question as to whether the inequalities obtained by this very trivial argument are in fact of a trivially obvious character, we consider their application to a hypothetical situation involving change of polarization on scattering in the forward direction. We consider only the case of spin-$\frac{1}{2}$ particles in order to avoid the complication of description involved for polarization of higher spins. Denoting the polarization of the incident beam of spin-$\frac{1}{2}$ particles by P_0 and that of the target of spin-$\frac{1}{2}$ particles by P_t (the axis again being the direction of $\mathbf{p}$), we may calculate the final polarization P_f by using matrix of Fig. 1(a) in the form

$$P_f=(\beta^2P_0-\gamma^2P_t)/(\beta^2+\gamma^2),$$

if we *assume* that the f matrix is pure imaginary. Applying the inequality (3) tells us that P_f lies between $\frac{1}{2}(P_0-P_t)$ and P_0. Thus, for an unpolarized target, P_f lies between $\frac{1}{2}P_0$ and P_0. One can similarly show, by the use of the matrix in Fig. 1(b), that for spin-$\frac{1}{2}$ particles incident on an unpolarized spin-1 target, P_f will lie between $\frac{1}{3}P_0$ and P_0. If the f matrix were not pure imaginary, or if the inequality (3) did not hold, P_f could lie anywhere between P_0 and $-P_t$ in the first instance, or between $-\frac{1}{3}P_0$ and P_0 in the second. Thus, these restrictions are not entirely of a trivial character.

GENERALIZATIONS

We have restricted our discussion so far to the particular case of elastic scattering with the possibility of spin flip, but the simple technique which is employed admits of a great variety of generalizations to other situations. It can, in fact, be applied to any (finite[2]) submatrix of the T matrix obtained by limiting the initial (and final) states to a (finite) subset. We illustrate with some examples.

Consider first elastic scattering of spinless particles. Let the initial momentum of one particle in the barycentric reference frame be $\mathbf{p}$ and the final momentum be $\mathbf{p}'$. If we restrict $\mathbf{p}$ and $\mathbf{p}'$ to a finite set of momenta $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$, all of the same magnitude, then the matrix $\langle \mathbf{p}_j | T | \mathbf{p}_k \rangle$ is a finite matrix of n rows and columns whose anti-Hermitian part must have positive-diagonal elements by the optical theorem. Application of the technique above then yields a set of inequalities satisfied by the matrix elements, which, in the case of rotational invariance, impose conditions on the imaginary part of the scattering amplitude at the angles defined by the initially selected set of momenta. Of course, in this case, since the T matrix can be immediately diagonalized by using a partial-wave expansion, the information obtained can be no more than one obtains from the fact that the imaginary part of each phase shift is positive.

However, if the particles have spin and the initial and final states are specified not only by the momentum of one particle but by the helicities of the particles as well, then a complete diagonalization is no longer trivally possible without knowledge of the dynamics of the situation, and the information then obtained is no longer so trivial.

As a second example, consider the reactions

$$\begin{Bmatrix}\pi+N\\K+\Lambda\end{Bmatrix}\rightarrow\begin{Bmatrix}\pi+N\\K+\Lambda\end{Bmatrix},$$

where the pion has momentum $\mathbf{p}$ in the barycentric frame and the kaon has momentum $\mathbf{p}'$ in this frame. Limiting oneself to one substate of the $I=\frac{1}{2}$ isospin state, and neglecting the ordinary spin of the baryon for the sake of the present argument, the T matrix within this subspace is a 2×2 matrix whose anti-Hermitian part can be written as

	(πN)	$(K\Lambda)$
(πN)	α	γ
$(K\Lambda)$	γ	β

The elements α and β are then related to the total cross sections for $\pi-N$ scattering and $K-\Lambda$ scattering, while γ is the imaginary part of the transition amplitude $\pi+N\rightarrow K+\Lambda$, with π and K having the specified momenta. The inequality

$$\alpha\beta-\gamma^2\geqslant 0,$$

would then set a lower bound on the $K-\Lambda$ total cross section (which is not directly measureable) at the given energy if one had knowledge of the imaginary part of the transition amplitude and the total $\pi-N$ cross section, both of which are in principle, at least, measureable. Numerous other examples involving many open channels can obviously be constructed.

As in the earlier example, practical application of these inequalities is severely limited by the fact that they apply only to a part (the anti-Hermitian, or in some cases, imaginary part) of the T matrix. This makes them difficult to apply profitably in a direct way to experimental results, but it is possible that they have useful applications to theory, especially in combination with other information such as that contained in dispersion relations.

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[2] The same methods can be applied to infinite matrices, but yield then the same information as is obtained from the totality of conditions derived from all its finite submatrices.